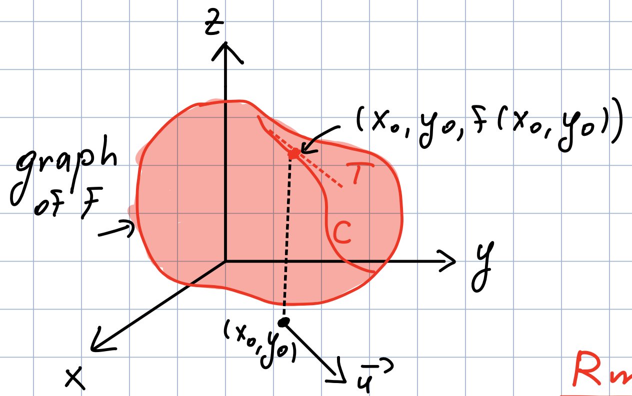
Last time: Directional derivatives, gradient

$f(x,y)$ $\qquad$ $\vec{u} = \langle a, b \rangle$ ← unit vector

$$\underbrace{D_{\vec{u}} f(x_0, y_0)}_{\text{directional derivative of } f \text{ at } (x_0,y_0) \text{ in the direction of } \vec{u}} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$



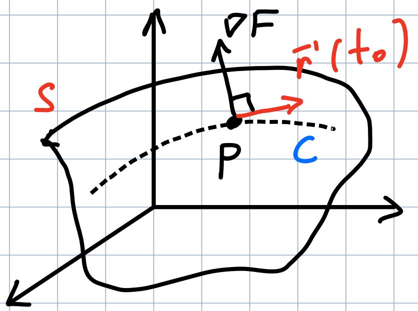
$$\boxed{D_{\vec{u}} f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b}$$

Rmk: $\vec{u} = \langle 1, 0 \rangle \Rightarrow D_{\vec{u}} f = f_x$; $\vec{u} = \langle 0, 1 \rangle \Rightarrow D_{\vec{u}} f = f_y$

Gradient: $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$ | $D_{\vec{u}} f(x,y) = \nabla f \cdot \vec{u}$

Tangent planes to level surfaces

$F(x,y,z)$



$S: F(x,y,z)=k$ – level surface

$P(x_0,y_0,z_0)$ point on S

A curve C on S through P is given by

$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$$

such that (1) $F(x(t), y(t), z(t)) = k$ $\quad (*)$

(2) $\vec{r}(t_0) = \langle x_0, y_0, z_0\rangle$

some value of parameter

$$\frac{d}{dt}(*): \quad 0 = \frac{dF}{dt} \underset{\text{Chain rule}}{=} \frac{\partial F}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial F}{\partial z}\cdot\frac{dz}{dt}$$

$$= \langle F_x, F_y, F_z\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle$$

$$\boxed{= \nabla F \cdot \vec{r}\,'(t)}$$

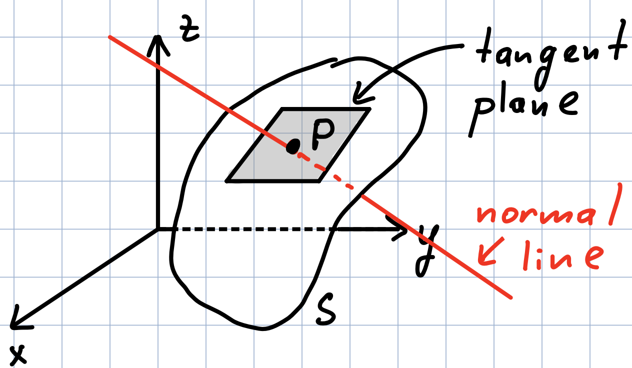
$\Rightarrow$ Gradient vector at P, $\nabla F(x_0,y_0,z_0)$, is orthogonal to the tangent vector to any curve C on S passing through P

(In particular, for $t=t_0$ $\quad \nabla F(x_0,y_0,z_0)\cdot \vec{r}\,'(t_0) = 0$

So: if $\nabla F(x_0, y_0, z_0) \neq 0$, then
the tangent plane to level surface $F(x,y,z) = k$ at $P(x_0, y_0, z_0)$
is the plane through P with normal vector $\nabla F(x_0, y_0, z_0)$.

I.e. tangent plane:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$



normal line to S at P:

line through P and orthogonal to tangent plane
(i.e. it's direction vector is $\nabla F(x_0, y_0, z_0)$)

$$\Longrightarrow \quad \frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Ex: S: $z = f(x,y)$ $\iff$ $F(x,y,z) = f(x,y) - z = 0$
-graph of f

Then $\nabla F = \langle f_x, f_y, -1 \rangle$

tangent plane: $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$

// f_x, f_y – "slopes" of the plane //

Ex: $\underbrace{\frac{x^2}{4} + y^2 + \frac{z^2}{9}}_{F(x,y,z)} = \underbrace{3}_{k}$ - ellipsoid. Find tangent plane, normal line at $P(-2, 1, -3)$

Sol: $\nabla F = \langle \frac{x}{2}, 2y, \frac{2}{9}z \rangle$ $\qquad \nabla F(-2, 1, -3) = \langle -1, 2, -\frac{2}{3} \rangle$

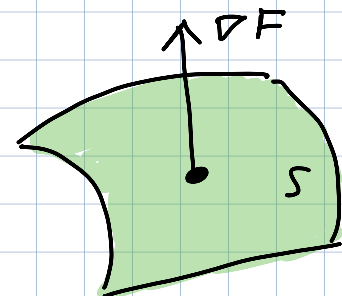
tangent plane: $-(x+2) + 2(y-1) - \frac{2}{3}(z+3) = 0$

normal line: $\dfrac{x+2}{-1} = \dfrac{y-1}{2} = \dfrac{z+3}{-2/3}$

Gradient summary:

- for $F(x,y,z)$

$\nabla F(x_0, y_0, z_0)$ <
- direction of max. increase of F
- perpendic. to level surface S



- for $f(x,y)$

$\nabla f(x_0, y_0)$ <
- dir. of max. increase of f
- perp. to level curve C